A new relationship for calculating the exponential integral used for constant-terminal-rate solution of diffusivity equation

Sayed Gomaa[1,2], PhD

[1]Mining and Petroleum Engineering Department, Faculty of Engineering

Al-Azhar University, Cairo, Egypt

[2]Petroleum Engineering and Gas Technology Department, The British University in Egypt

Email: Sayed.gomaa@bue.edu.eg

Mobile: 002-01115003694

Abstract

The constant-terminal-rate solution of diffusivity equation is an integral part of most transient test analysis techniques, such as with drawdown and pressure buildup analyses. Most of these tests involve producing the well at a constant flow rate and recording the flowing pressure as a function of time. There are two commonly used forms of the constant-terminal-rate solution: the E_i-function solution and the dimensionless pressure solution.

This paper presents a new relationship for calculating the exponential integral E_i with an average error of 0.026% and correlation coefficient of 0.999999999 for the range $0.01 < x \leq 3.0$.

Another new relationship was developed for calculating the exponential integral E_i with an average error of 6.05 and correlation coefficient of 0.999988 for the range $3.0 < x \leq 9.8$.

Introduction

To obtain a solution to the diffusivity equation it is necessary to specify an initial condition and impose two boundary conditions.

The initial condition simply states that the reservoir is at a uniform pressure pi when production begins. The two boundary conditions require that the well is producing at a constant production rate and that the reservoir behaves as if it were infinite in size, i.e., re = ∞.

Based on the boundary conditions, there are two generalized solutions to the diffusivity equation:

• Constant-terminal-pressure solution

• Constant-terminal-rate solution

Constant-terminal-pressure solution

In the constant-rate solution to the radial diffusivity equation, the flow rate is considered to be constant at a certain radius (usually wellbore radius) and the pressure profile around that radius is determined as a function of time and position. In the constant-terminal-pressure solution, the pressure is known to be constant at some particular radius and the solution is designed to provide the cumulative fluid movement across the specified radius (boundary). The constant-pressure solution is widely used in water influx calculations [1].

Constant-terminal-rate solution

The constant-terminal-rate solution is an integral part of most transient test analysis techniques, such as with drawdown and pressure buildup analyses. Most of these tests involve producing the well at a constant flow rate and recording the flowing pressure as a function of time. There are two commonly used forms of the constant-terminal-rate solution [2]:

• The E_i-function solution

• The dimensionless pressure solution

The first form of solution is discussed below.

The E_i-Function Solution

Matthews and Russell [3] proposed a solution to the diffusivity equation that is based on the following assumptions:

• Infinite acting reservoir, i.e., the reservoir is infinite in size

• The well is producing at a constant flow rate

• The reservoir is at a uniform pressure, p_i, when production begins

• The well, with a wellbore radius of r_w, is centered in a cylindrical reservoir of radius r_e.

• No flow across the outer boundary, i.e., at r_e.

Employing the above conditions, the authors presented their solution in the following form:

$$p(r,t) = p_i + \left[\frac{70.6 q_o B_o \mu_o}{kh}\right] E_i\left[\frac{-948\phi\mu_o c_t r^2}{kt}\right] \qquad (1)$$

$p(r,t)$ = pressure at radius r from the well after t hours

t = time, hrs

k = permeability, md

q_o = flow rate, STB/day

The mathematical function, E_i, is called the exponential integral and is defined by:

$$E_i(-x) = -\int_x^\infty \frac{e^{-u}du}{u} = \left[\ln x - \frac{x}{1!} + \frac{x^2}{2(2!)} - \frac{x^3}{3(3!)} + etc.\right] \qquad (2)$$

Craft, Hawkins, Terry and Rogers [4] presented the values of the E_i-function in tabulated and graphical forms as shown in Table a1 and Figure a1, respectively in the appendix. The E_i solution, as expressed by Equation 1, is commonly referred to as the line-source solution.

The exponential integral E_i can be approximated by the following equation when its argument x is less than 0.01:

For $x < 0.02$

$$-E_i(-x) = \ln(x) + 0.577 \quad (3)$$

For $0.02 < x < 0.01$

$$E_i(-x) = \ln(1.781x) \quad (4)$$

where the argument x in this case is given by:

$$x = \frac{948\phi\mu_o c_t r^2}{kt}$$

Another expression that can be used to approximate the E_i-function for the range 0.01 < x < 3.0 is given by [3]:

$$E_i(-x) = a_1 + a_2\ln(x) + a_3[\ln(x)]^2 + a_4[\ln(x)]^3 + a_5x + a_6x^2 + a_7x^3 + a_8/x \quad (5)$$

With the coefficients a_1 through a_8 having the following values:

$a_1 = -0.33153973$ $a_2 = -0.81512322$ $a_3 = 5.22123384 \times 10^{-2}$

$a_4 = 5.9849819 \times 10^{-3}$ $a_5 = 0.66231845$ $a_6 = -0.12333524$

$a_7 = 1.0832566 \times 10^{-2}$ $a_8 = 8.6709776 \times 10^{-4}$

Proposed relationships

Two relationships for calculating E_i were developed by use of the linear and nonlinear regression analysis.

Proposed relationship I

For $0.01 < x \leq 3.0$

Equation 4 is used with new eight coefficients a_1 through a_8 having the following values:

$a_1 = 0.340114$ $a_2 = 0.1527766$ $a_3 = 0.279658$

$a_4 = 0.055861$ $a_5 = 0.126038$ $a_6 = 0.032155$

$a_7 = 0.003158$ $a_8 = 0.034319$

Equation 4 with new eight coefficients a_1 through a_8 approximate the E_i-values with an average error of 0.026%.

Proposed relationship II

For $3 < x \leq 9.8$

$$E_i(x) = ab^x x^c$$

Where

$a = 0.636451$ $b = 0.357482$ $c = -0.729568$

Statistical Error analysis

The statistical and graphic error analyses were used to check the performance, as well as the accuracy, of the $E_i$correlations developed in this study and by Ahmed study.

The accuracy of correlations relative to the tabulated values after Craft is determined by various statistical means. The criteria used in this study were average percent relative error, average absolute percent relative error, minimum/maximum absolute percent relative error, standard deviation, and the correlation coefficient.

Average Relative Error

This is an indication of the relative deviation in percent from the experimental values and is given by:

$$\left(\sum_{j=1}^{n} RD_j\right)/n$$

RD_i is the relative deviation in percent of an estimated value from an experimental value and is defined by:

$$RD_j = \left[\frac{\left(E_{i_{cal}} - E_{i_{tab}}\right)}{E_{i_{tab}}}\right]_j \times 100$$

where x_{cal} and x_{tab} represent the calculated and tabulated values, respectively. The lower the value of E_r the more equally distributed are the errors between positive and negative values.

Average Absolute Relative Error

This is defined as:

$$\left(\sum_{i=1}^{n} RD_j\right)/n$$

and indicates the relative absolute deviation in percent from the tabulated values. A lower value implies a better correlation.

Minimum/Maximum Absolute Relative Error

After the absolute percent relative error for each data point is calculated, $|RD_j|, j = 1,2, \dots n$, both the minimum and maximum values are scanned to know the range of error for each correlation:

$$RD_{min} = \min_{j=1\ to\ n} |RD_j|$$

and

$$RD_{max} = \max_{J=1\ to\ n} |RD_j|$$

The accuracy of a correlation can be examined by maximum absolute percent relative error. The lower the value of maximum absolute percent relative error, the higher the accuracy of the correlation is.

Standard Deviation

Standard deviation s_x is a measure of dispersion and is expressed as:

$$s^2{}_x = \left(\sum_{j=1}^{n} RD_j{}^2\right) / (n-1)$$

A lower value of standard deviation means a smaller degree of scatter.

Correlation Coefficient

The correlation coefficient, r, represents the degree of success in reducing the standard deviation by regression analysis. It is defined as:

$$\sum_{i=1}^{n} \left(E_{i_{cal}} - E_{i_{tab}}\right)^2$$

$$r^2 = 1 - \left[\sum_{i=1}^{n} \left(E_{i_{cal}} - E_{i_{tab}}\right)^2 / \sum_{i=1}^{n} \left(E_{i_{cal}} - E_{i_{avg}}\right)^2\right]$$

where

$$E_{i_{avg}} = \left(\sum_{i=1}^{n} E_{i_{tab}}\right)/n$$

The correlation coefficient lies between 0 and 1. A value of 1 indicates a perfect correlation, whereas a value of 0 implies no correlation at all among the given independent variables.

Crossplot

In this technique, all the calculated values are plotted vs. the tabulated values, and thus a crossplot is formed. A 45° straight line is drawn on the crossplot on which estimated value is equal to experimental value. The closer the plotted data points are to this line, the better the correlation is.

Comparison of Correlations

Statistical Error Analysis

Average percent relative error, average absolute percent relative error, minimum/maximum percent relative error, minimum/maximum absolute percent relative error, standard deviation, and correlation coefficient were computed for each correlation.

Table 1 presents the comparison of errors relative to the tabulated $E_i(-x)$ calculated from two correlations. The correlation for $E_i(-x)$ of this study achieved the lowest errors and standard deviation, with the highest correlation coefficient accuracy of 0.999999999, as presented in Table 2.

Table 1 Comparison of $-E_i(-x)$ calculated by correlations from this study and Ahmed

Values of $-E_i(-x)$	Deviation of calculated values, % of Craft values

x	After Craft	Ahmed Study	This Study	Ahmed Study	This study
0.1	1.82292	1.822791	1.822926	0.007101	0.000333
0.2	1.22265	1.222597	1.222641	0.004321	0.000754
0.3	0.90568	0.905864	0.905718	0.02031	0.004212
0.4	0.70238	0.702637	0.702365	0.03666	0.002162
0.5	0.55977	0.559967	0.559745	0.035147	0.004447
0.6	0.45438	0.454448	0.454367	0.014996	0.002917
0.7	0.37377	0.373708	0.373776	0.0166	0.001586
0.8	0.3106	0.310433	0.310616	0.05363	0.004994
0.9	0.26018	0.25996	0.260204	0.084512	0.00937
1	0.21938	0.219143	0.219398	0.107966	0.008205
1.1	0.18599	0.185771	0.185995	0.117598	0.002505
1.2	0.15841	0.158238	0.158402	0.108417	0.005286
1.3	0.13545	0.135347	0.135436	0.076216	0.010345
1.4	0.11622	0.116186	0.116200	0.029474	0.017171
1.5	0.10002	0.10005	0.100000	0.030365	0.019769
1.6	0.08631	0.086388	0.086293	0.090496	0.020179
1.7	0.07465	0.074761	0.074645	0.149059	0.006593
1.8	0.06471	0.06482	0.064711	0.170612	0.001375
1.9	0.0562	0.056285	0.056209	0.151054	0.016188
2	0.0489	0.048929	0.04891	0.058583	0.021369
2.1	0.04261	0.042569	0.042627	0.095559	0.039193
2.2	0.03719	0.03706	0.037203	0.349947	0.033643
2.3	0.0325	0.032282	0.032509	0.669366	0.028346
2.4	0.02844	0.028143	0.02844	1.043656	0.000223
2.5	0.02491	0.024568	0.024906	1.372316	0.016887
2.6	0.02185	0.0215	0.021833	1.600417	0.078617
2.7	0.01918	0.018897	0.01916	1.476464	0.106642
2.8	0.01686	0.016727	0.016835	0.789041	0.148321
2.9	0.01482	0.01497	0.014817	1.015219	0.019683
3	0.01305	0.013616	0.013071	4.336522	0.163226

Table 2 Statistical accuracy of E_i correlations

	Ahmed study	This study
ARE, %	-0.06312	-0.00418
AARE, %	0.470387	0.026485

Min. ARE, %	-1.60042	-0.14832
Max. ARE, %	4.336522	0.163226
Min. AARE, %	0.004321	0.000223
Max. AARE, %	4.336522	0.163226
Standard deviation, %	0.981432	0.050025
Correlation coefficient	0.999999789	0.999999999

Crossplot

The crossplot of calculated values of E_i from this study's correlation vs. tabulated values for E_i after Craft is presented in Figs. 1. The plotted points of this study's correlation fall very close to the perfect correlation of the 45° line.

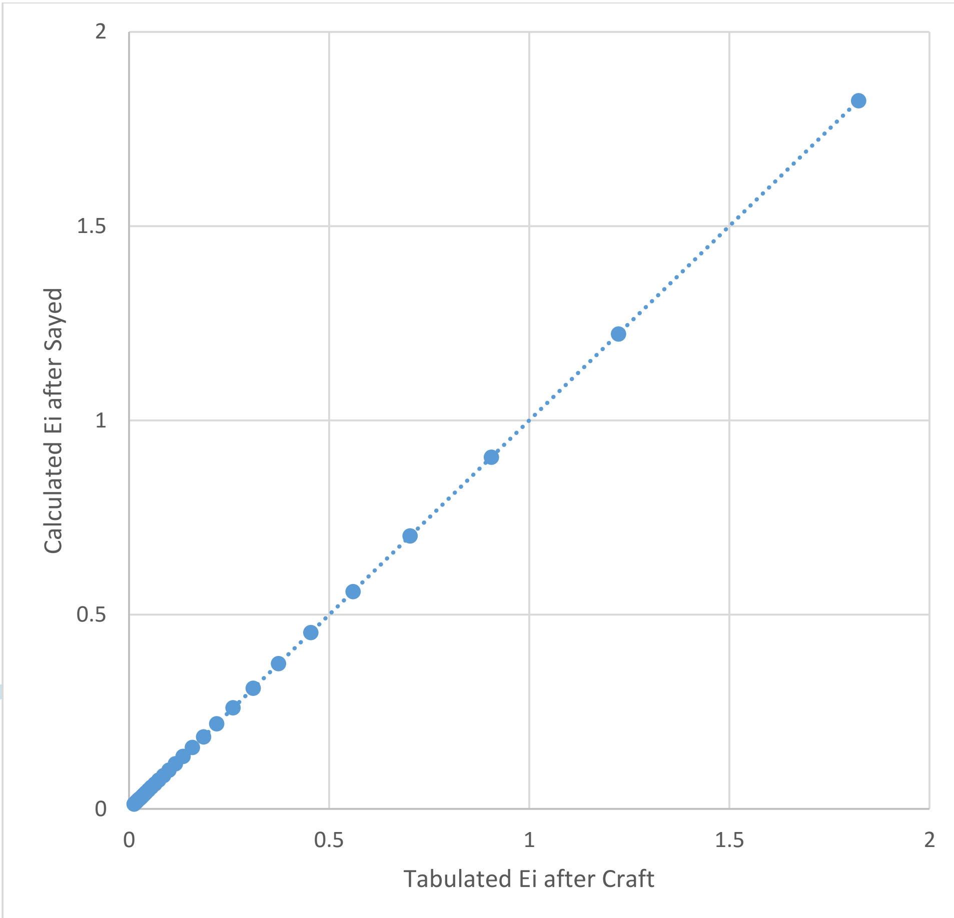

Figure 1 Crossplot for E_i from correlation I (this study's correlation).

Proposed correlation II

For $3 < x \leq 9.8$

$$E_i(x) = ab^x x^c$$

Where

$a = 0.636451$ $b = 0.357482$ $c = -0.729568$

Table 2 Values of $-E_i(-x)$ after Craft and from this study

x	After Craft	This study	AARE, %
3.1	0.01149	0.011491	0.011774
3.2	0.01013	0.010131	0.006365
3.3	0.00894	0.008937	0.028763
3.4	0.00789	0.00789	0.000499
3.5	0.00697	0.00697	0.002950

3.6	0.00616	0.006161	0.008935
3.7	0.00545	0.005448	0.030588
3.8	0.00482	0.004821	0.021245
3.9	0.00427	0.004268	0.044528
4	0.00378	0.00378	0.010635
4.1	0.00335	0.00335	0.001196
4.3	0.00263	0.002634	0.148369
4.4	0.00234	0.002337	0.132306
4.5	0.00207	0.002074	0.201791
4.6	0.00184	0.001842	0.089719
4.7	0.00164	0.001636	0.25878
4.8	0.00145	0.001453	0.231729
4.9	0.00129	0.001292	0.132663
5	0.00115	0.001148	0.139927
5.1	0.00102	0.001021	0.124545
5.2	0.00091	0.000908	0.167445
5.3	0.00081	0.000808	0.202743
5.4	0.00072	0.000719	0.075038
5.5	0.00064	0.00064	0.077871
5.6	0.00057	0.00057	0.059765
5.7	0.00051	0.000508	0.394915
5.8	0.00045	0.000453	0.566668
5.9	0.0004	0.000403	0.812800
6	0.00036	0.000359	0.167190
6.1	0.00032	0.00032	0.118416

Table 2 Values of $-E_i(-x)$ after Craft and from this study

x	After Craft	This study	AARE, %
6.2	0.00029	0.000286	1.499312
6.3	0.00026	0.000255	2.024009
6.4	0.00023	0.000227	1.212778
6.5	0.0002	0.000203	1.347157
6.6	0.00018	0.000181	0.47477
6.7	0.00016	0.000161	0.871911
6.8	0.00014	0.000144	2.894815
6.9	0.00013	0.000129	1.081391
7	0.00012	0.000115	4.323233
7.1	0.0001	0.000103	2.522407
7.2	0.00009	9.16E-05	1.734955
7.3	0.00008	8.18E-05	2.229859
7.4	0.00007	7.31E-05	4.372049
7.5	0.00007	6.53E-05	6.748338
7.6	0.00006	5.83E-05	2.78527
7.7	0.00005	5.21E-05	4.254997

7.8	0.00005	4.66E-05	6.8176
7.9	0.00004	4.16E-05	4.119721
8	0.00004	3.72E-05	6.916467
8.1	0.00003	3.33E-05	10.96882
8.2	0.00003	2.98E-05	0.770994
8.3	0.00003	2.66E-05	11.25918
8.5	0.00002	2.13E-05	6.493032
8.6	0.00002	1.91E-05	4.733381
8.7	0.00002	1.7E-05	14.76791
8.8	0.00002	1.53E-05	23.73814
8.9	0.00001	1.36E-05	36.48427
9	0.00001	1.22E-05	22.14278
9.1	0.00001	1.09E-05	9.318104
9.2	0.00001	9.78E-06	2.15139
9.3	0.00001	8.76E-06	12.40997
9.4	0.00001	7.84E-06	21.58641
9.5	0.00001	7.02E-06	29.79568
9.6	0.00001	6.29E-06	37.14043
9.7	0.00001	5.63E-06	43.71231
9.8	0.00001	5.04E-06	49.5932

Table 3 Statistical accuracy of proposed correlation II

ARE, %	-2.63425
AARE, %	6.054048
Min. ARE, %	-49.5932
Max. ARE, %	36.48427
Min. AARE, %	0.000499
Max. AARE, %	49.5932
Standard deviation, %	12.63
Correlation coefficient	0.999988

Cross plot

The crossplot of calculated values of E_i from this study's correlation vs. tabulated values for E_i after Craft was presented in Figs. 2. The plotted points of this study's correlation fall very close to the perfect correlation of the 45° line.

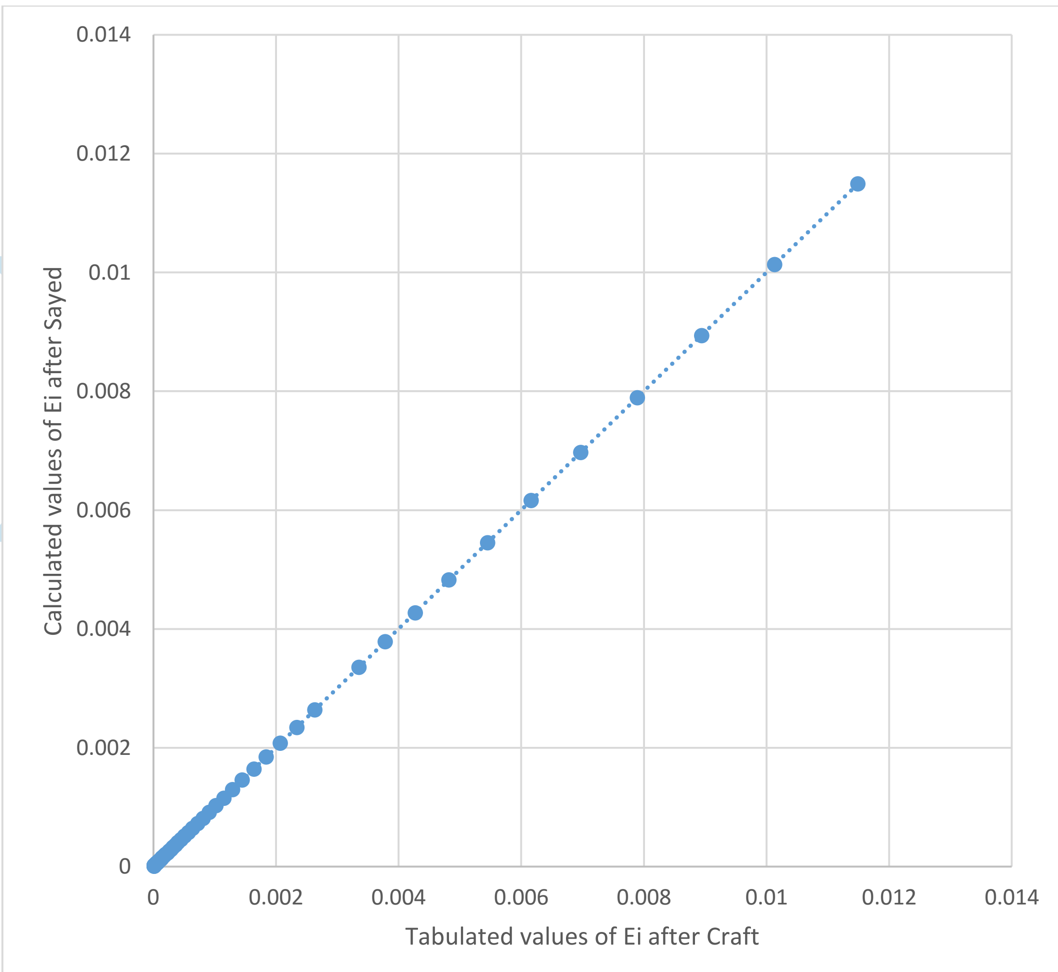

Figure 2 Crossplot for E_i from correlation II (this study's correlation).

Conclusions

From this paper, one may conclude that:

1. This paper presents two new correlations for calculating the exponential integral, E_i used for constant-terminal-rate solution of diffusivity equation.

2. Deviations from tabulated values of E_i after Craft, indicated as average percent relative errors, average absolute percent relative errors, and the standard deviations, were lower for this study than for calculated values based on Ahmed correlation.

3. The correlation coefficient of the correlations of this study are closer to one than that of Ahmed correlation.

Nomenclature

h= thickness, ft

k = permeability, md

$p(r,t)$ = pressure at radius r from the well after t hours, psi

p_i = initial reservoir pressure, psi

t = time, hrs

q_o = flow rate, STB/day

B_o= oil formation volume factor, bbl/STB

μ_o= oil viscosity, cp

ϕ= porosity, fraction

c_t= total compressibility, psi^{-1}

RD_i = Relative deviation, %

ARE = Average Relative Error, %

AARE = Average Absolute Relative Error, %

Min. ARE = Minimum Absolute Relative Error, %

Max. ARE = Maximum Absolute Relative Error, %

S_x = Standard Deviation

R^2 = Correlation Coefficient

References

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2. Dake, L. P., The Practice of Reservoir Engineering. Amsterdam, Elsevier Revised Edition 2001.

3. Ahmed, T, Advanced Reservoir Management and Engineering, Elsevier publications, second edition 2012.

4. Craft B., Hawkins M., Terry R. and Rogers J., Applied Petroleum Reservoir Engineering, 3rd edition, Prentice Hall, 2015.

Appendix

Table a1 Values of the −Ei (−x) as a Function of x (After Craft, Hawkins, and Terry, 1991)

x	$-E_i(-x)$	x	$-E_i(-x)$	x	$-E_i(-x)$
0.1	1.82292	3.3	0.00894	6.6	0.00018
0.2	1.22265	3.4	0.00789	6.7	0.00016
0.3	0.90568	3.5	0.00697	6.8	0.00014
0.4	0.70238	3.6	0.00616	6.9	0.00013

0.5	0.55977	3.7	0.00545	7	0.00012
0.6	0.45438	3.8	0.00482	7.1	0.0001
0.7	0.37377	3.9	0.00427	7.2	0.00009
0.8	0.3106	4	0.00378	7.3	0.00008
0.9	0.26018	4.1	0.00335	7.4	0.00007
1	0.21938	4.3	0.00263	7.5	0.00007
1.1	0.18599	4.4	0.00234	7.6	0.00006
1.2	0.15841	4.5	0.00207	7.7	0.00005
1.3	0.13545	4.6	0.00184	7.8	0.00005
1.4	0.11622	4.7	0.00164	7.9	0.00004
1.5	0.10002	4.8	0.00145	8	0.00004
1.6	0.08631	4.9	0.00129	8.1	0.00003
1.7	0.07465	5	0.00115	8.2	0.00003
1.8	0.06471	5.1	0.00102	8.3	0.00003
1.9	0.0562	5.2	0.00091	8.5	0.00002
2	0.0489	5.3	0.00081	8.6	0.00002
2.1	0.04261	5.4	0.00072	8.7	0.00002
2.2	0.03719	5.5	0.00064	8.8	0.00002
2.3	0.0325	5.6	0.00057	8.9	0.00001
2.4	0.02844	5.7	0.00051	9	0.00001
2.5	0.02491	5.8	0.00045	9.1	0.00001
2.6	0.02185	5.9	0.0004	9.2	0.00001
2.7	0.01918	6	0.00036	9.3	0.00001
2.8	0.01686	6.1	0.00032	9.4	0.00001
2.9	0.01482	6.2	0.00029	9.5	0.00001
3	0.01305	6.3	0.00026	9.6	0.00001
3.1	0.01149	6.4	0.00023	9.7	0.00001
3.2	0.01013	6.5	0.0002	9.8	0.00001

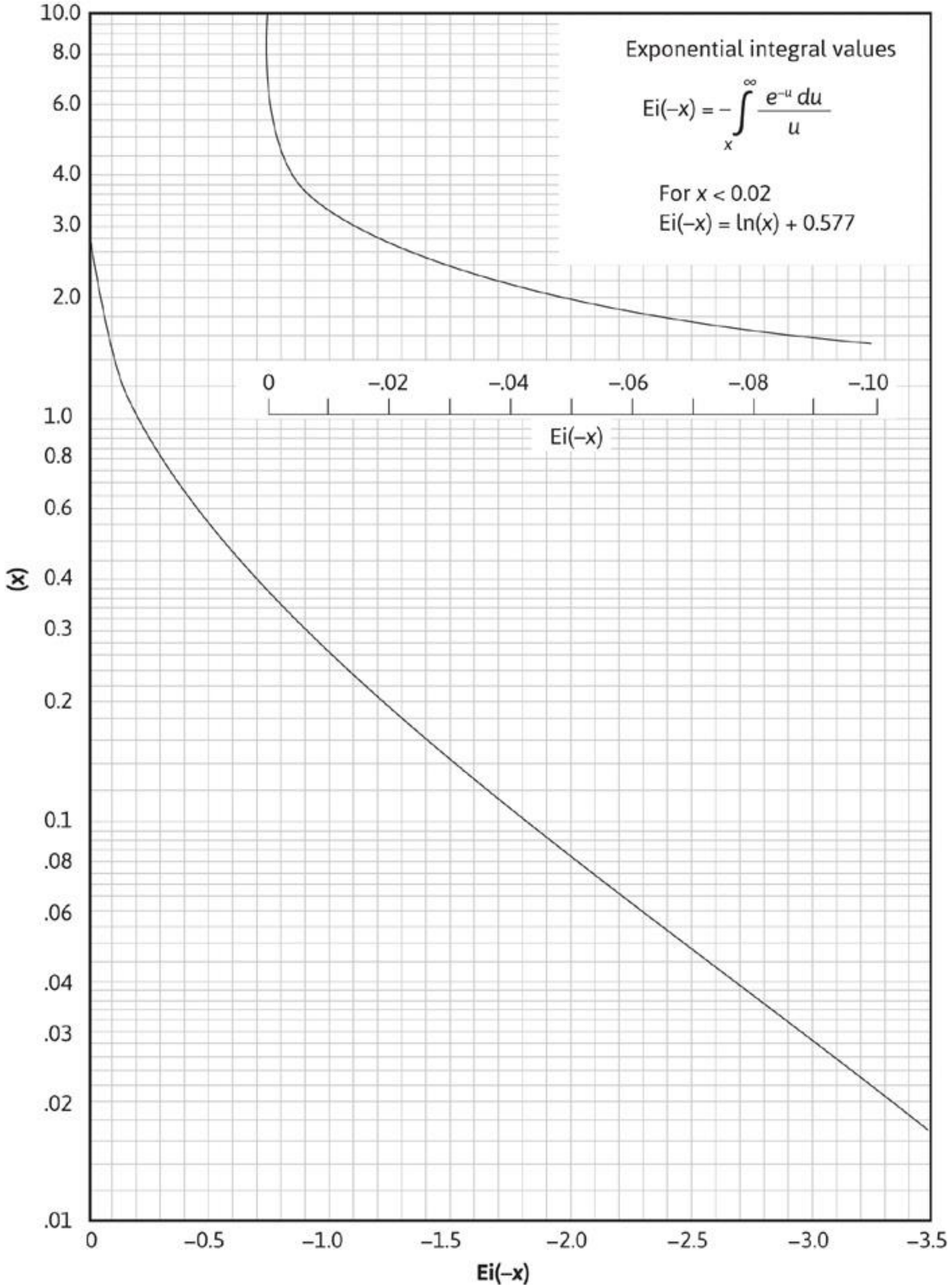

Figure a.1 Plot of exponential integral function. (After Craft, Hawkins, Terry and Rogers, 2015)