Investigation of Fractional Partial Differential Equations in the College Course Mathematical Physics Equations

Qinghua Feng
Shandong University of Technology
School of Mathematics and Statistics
Zhangzhou Road 12, Zibo, 255049, China
fqhua@sina.com

Abstract: We introduce a new approach for solving fractional partial differential equations, where the fractional derivative is defined in the sense of the conformable fractional derivative. As for applications of this approach, we apply it to seek exact solutions for the space-time fractional BBM equation successfully.

Key–Words: Fractional partial differential equation; Exact solution; Space-time fractional BBM equation

1 Introduction

In the teaching of the college course Mathematical Physics Equations, fractional partial differential equations is an important topic. Recently, Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics. In particular, fractional derivative is useful in describing the memory and hereditary properties of materials and processes. To illustrate better the physical phenomena denoted by fractional differential equations, it is necessary to obtain analytical or numerical solutions for fractional differential equations. Many efficient methods have been proposed so far [1-12].

In this paper, we introduce a new approach to seek exact solutions for space-time fractional partial differential equations based on the combination of the simple equation method and the following Jacobi elliptic equation

$$(G')^2 = e_2 G^4 + e_1 G^2 + e_0, \tag{1}$$

where $e_0,\ e_1,\ e_2$ are arbitrary constants. The fractional partial differential equations are defined in the sense of the conformable fractional derivative, which is defined as follows

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

and satisfies the following properties:

$(i).\ D_t^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}.$

$(ii).\ D_t^\alpha f[g(t)] = f'_g[g(t)] D_t^\alpha g(t).$

2 Applications of the present method to Space-time fractional BBM equation

Consider the space-time fractional BBM equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+u\frac{\partial^\beta u}{\partial x^\beta}+\frac{\partial^\beta u}{\partial x^\beta}-\mu\frac{\partial^{2\beta+\alpha}u}{\partial x^\beta\partial x^\beta t^\alpha}=0,\ 0<\alpha,\ \beta\leq 1, \tag{2}$$

which is a variation of the following BBM equation of integer order:

$$u_t+uu_x+u_x-\mu u_{xxt}=0. \tag{3}$$

In order to apply the present method described in Section 2, suppose $u(x,t)=U(\xi)$, where $\xi=\frac{ct^\alpha}{\alpha}+\frac{kx^\beta}{\beta}+\xi_0$, $k,\ c,\ \xi_0$ are all constants with $k,\ c\neq 0$. Then by use of the properties (i) and (ii), we obtain

$$\begin{cases} D_t^\alpha u=D_t^\alpha U(\xi)=U'(\xi)D_t^\alpha\xi=cU'(\xi),\\ D_x^\beta u=D_x^\beta U(\xi)=U'(\xi)D_x^\beta\xi=kU'(\xi),\end{cases} \tag{4}$$

and then Eq. (2) can be turned into the following form:

$$cU'+kUU'+kU'-\mu ck^2U'''=0. \tag{5}$$

Suppose that the solution of Eq. (5) can be expressed by

$$U(\xi)=\sum_{i=0}^{n}a_i(\frac{G'}{G})^i, \tag{6}$$

where $G=G(\xi)$ satisfies the Jacobo elliptic equation (1). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (5), we can obtain $n=2$. So we have

$$U(\xi)=a_0+a_1(\frac{G'}{G})+a_2(\frac{G'}{G})^2. \tag{7}$$

Substituting (7) into (5), using Eq. (1), and collecting all the terms with the same power of $G^iG'^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields that

$$a_0=-\frac{k+c+8\mu ck^2e_1}{k},\ a_1=0,\ a_2=12\mu ck.$$

Substituting the result above into Eq. (7), and combining with (11) we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for Eq. (2), where $\xi=\frac{ct^\alpha}{\alpha}+\frac{kx^\beta}{\beta}+\xi_0$.

Family 1: when $e_2=m^2,\ e_1=-(1+m^2),\ e_0=0$,

$$u_1(x,t)=-\frac{k+c+8\mu ck^2e_1}{k}+12\mu ck[cn(\xi)ds(\xi)]^2, \tag{8}$$

Family 2: when $e_2=-m^2,\ e_1=2m^2-1,\ e_0=1-m^2$,

$$u_2(x,t)=-\frac{k+c+8\mu ck^2e_1}{k}+12\mu ck[sn(\xi)dc(\xi)]^2, \tag{9}$$

Family 3: when $e_2=-1,\ e_1=2-m^2,\ e_0=m^2-1$,

$$u_3(x,t)=-\frac{k+c+8\mu ck^2e_1}{k}+12\mu ckm^4[sn(\xi)cd(\xi)]^2. \tag{10}$$

Family 4: when $e_2 = 1,\ e_1 = 2 - m^2,\ e_0 = 1 - m^2,$

$$u_4(x,t) = -\frac{k + c + 8\mu ck^2 e_1}{k} + 12\mu ck[\frac{dc(\xi)}{sn(\xi)}]^2. \tag{11}$$

Family 5: when $e_2 = m^2(m^2 - 1),\ e_1 = 2m^2 - 1,\ e_0 = 1,$

$$u_5(x,t) = -\frac{k + c + 8\mu ck^2 e_1}{k} + 12\mu ck[\frac{cs(\xi)}{dn(\xi)}]^2. \tag{12}$$

Family 6: when $e_2 = 1,\ e_1 = -(m^2 + 1),\ e_0 = m^2,$

$$u_6(x,t) = -\frac{k + c + 8\mu ck^2 e_1}{k} + 12\mu ck(1 - m^2)^2[\frac{sd(\xi)}{cn(\xi)}]^2. \tag{13}$$

Remark. Combining with other general solutions of the Jacobi elliptic equation (1) where $e_2,\ e_1,\ e_0$ taken different values, one can obtain corresponding hyperbolic function solutions, trigonometric function solutions and rational function solutions for space-time fractional BBM equation, which are omitted here for the sake of simplicity.

3 Conclusions

We have introduced a new approach for solving fractional partial differential equations in the sense of the conformable fractional derivative, and apply it to seek exact solutions for the space-time fractional BBM equation. As a result, a series of exact solutions in various forms are successfully found.

References:

[1] *İ. Aslan, Traveling Wave Solutions for Nonlinear Differential-Difference Equations of Rational Types, Commun. Theor. Phys. 65 (2016) 39-45.*

[2] *B. Agheli, R. Darzi and A. Dabbaghian, Computing exact solutions for conformable time fractional generalized seventh-order KdV equation by using (G'/G)-expansion method, Opt. Quant. Electron. 49:387 (2017) 1-13.*

[3] *Q. Feng, Jacobi Elliptic Function Solutions For Fractional Partial Differential Equations, IAENG International Journal of Applied Mathematics,* **46(1)***(2016), 121-129.*

[4] *S. Zhang and H.Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, Phys. Lett. A,* **375***(2011), 1069-1073.*

[5] *E. M. E. Zayed, The (G'/G)-expansion method and its applications to some nonlinear evolution equations in the mathematical physics, J. Appl. Math. Computing,* **30***(2009), 89-103.*

[6] *Q. Feng and F. Meng, Explicit solutions for space-time fractional partial differential equations in mathematical physics by a new generalized fractional Jacobi elliptic equation-based sub-equation method, Optik* **127***(2016), 7450-7458.*

[7] *A.M.A. El-Sayed, S.H. Behiry and W.E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation, Comput. Math. Appl.,* **59***(2010), 1759-1765.*

[8] *O. Acan, O. Firat, Y. Keskin and G. Oturanc, Conformable variational iteration method, New Trends in Math. Sci.* **5(1)***(2017), 172-178.*

[9] *M. Yavuz and B. Yaşkıran, Approximate-analytical solutions of cable equation using conformable fractional operator, New Trends in Math. Sci.* **5(4)***(2017), 209-219.*

[10] *A. Kurt, O. Tasbozan and D. Baleanu, New solutions for conformable fractional Nizhnik-Novikov-Veselov system via G'/G expansion method and homotopy analysis methods, Opt. Quant. Electron. 49:333 (2017) 1-16.*

[11] *T. Islam, M. Ali Akbar and A. K. Azad, Traveling wave solutions to some nonlinear fractional partial differential equations through the rational (G'/G)-expansion method, J. Ocean Engi. Sci. 3 (2018) 76-81.*

[12] *S. Guo, L. Mei and Y. Li, Fractional variational homotopy perturbation iteration method and its application to a fractional diffusion equation, Appl. Math. Comput.,* **219***(2013), 5909-5917.*